Links between quantum and classical DFT and the problem of correlation in classical DFT

**Daniel Borgis** 

Pôle de Physico-Chimie Théorique Ecole Normale Supérieure de Paris

GdR Corrélation, 27-29 Novembre 2013







Electronic density functional theory (eDFT)



**David Mermin**, *Thermal properties of the inhomogeneous electron gas*, Phys. Rev. 137 (1965).

**R. Evans**, *The nature of the liquid-vapour interface and other topics in Statistical physics of non-uniform fluids*, Advances in Physics (1979).

**Y. Rosenfeld**, *Free-energy model for the inhomogeneous hard-sphere fluid mixture*, Phys. Rev. Lett. (1989).

And many others: J. Percus, JP Hansen, H. Löwen, ....

.....Mostly physicists !!

### Classical Density Functional Theory (cDFT)



Thermodynamic equilibrium

## Classical Density Functional Theory (cDFT)



Thermodynamic equilibrium

### Molecular Density Functional Theory (MDFT)



• Atomistic (or quantum mechanical) solute creating at each point in space an external potential





Thermodynamic equilibrium

### Molecular Density Functional Theory (MDFT)



Functional minimization:  $\frac{\partial \Theta[\rho]}{\delta \rho} = 0$ 

(- 0)

Thermodynamic equilibrium



 $F[\rho] = \Theta[\rho] - \Theta[\rho_0]$ 

 $F_{\min} =$ Solvation free energy

But what is the functional ??

The exact functional

$$F[\rho(\mathbf{x})] = F_{id}[\rho] + F_{exc}[\rho] + F_{ext}[\rho] \qquad \mathbf{x} = (\mathbf{r}, \mathbf{\Omega})$$

$$F_{id}[\rho] = k_B T \int d\mathbf{x}_1 \left[ \rho(\mathbf{x}_1) \ln\left(\frac{\rho(\mathbf{x}_1)}{\rho_0}\right) - \rho(\mathbf{x}_1) + \rho_0 \right]$$

$$F_{ext}[\rho] = \int d\mathbf{x}_1 \, V_{ext}(\mathbf{x}_1) \, \rho(\mathbf{x}_1)$$

$$ρ(\mathbf{r}, \Omega)$$

$$F_{exc}[\rho] = \frac{1}{2} \iint d\mathbf{x}_1 d\mathbf{x}_2 \ \Delta \rho(\mathbf{x}_1) \ u_{eff}(\mathbf{x}_1, \mathbf{x}_2; [\rho]) \ \Delta \rho(\mathbf{x}_2), \qquad \Delta \rho(\mathbf{x}) = \rho(\mathbf{x}) - \rho_0$$

 $u_{eff}(\mathbf{x}_1, \mathbf{x}_2; [\rho]) =$  Effective solvent-solvent pair potential

$$u_{eff}(\mathbf{x}_1, \mathbf{x}_2; [\rho]) = u(\mathbf{x}_1, \mathbf{x}_2) + u_{cor}(\mathbf{x}_1, \mathbf{x}_2; [\rho])$$

#### Fundamental Measure Theory for Hard-Sphere Mixtures



Y. ROSENFELD, Phys. Rev. Lett. 63, 980 (1989).

E. KIERLIK and M. L. ROSINBERG, Phys. Rev. A 42, 3382 (1990)

 $\Theta[\{\rho_i(\mathbf{r})\}] = \mathcal{F}[\{\rho_i(\mathbf{r})\}] + \Theta[\{\rho_i^0\}]$ 

$$\begin{aligned} \mathcal{F}[\{\rho_i(\mathbf{r})\}] &= \mathcal{F}_{id}[\{\rho_i(\mathbf{r})\}] + \mathcal{F}_{ext}[\{\rho_i(\mathbf{r})\}] \\ &+ \mathcal{F}_{exc}[\{\rho_i(\mathbf{r})\} - \mathcal{F}_{exc}[\{\rho_i^0\}] - \sum_i \mu_{exc}^i \int d\mathbf{r} \left(\rho_i(\mathbf{r}) - \rho_i^0\right) \\ \mathcal{F}_{id}[\{\rho_i(\mathbf{r})\}] &= k_B T \sum_i \int d\mathbf{r} \left[\rho_i(\mathbf{r}) \ln \left(\frac{\rho_i(\mathbf{r})}{\rho_i^0}\right) - \rho_i(\mathbf{r}) + \rho_i^0\right] \\ \mathcal{F}_{ext}[\{\rho_i(\mathbf{r})\}] &= \sum_i \int d\mathbf{r} V_i(\mathbf{r}) \rho_i(\mathbf{r}) \end{aligned}$$

#### Fundamental Measure Theory for Hard-Sphere Mixtures



E. KIERLIK and M. L. ROSINBERG, Phys. Rev. A 42, 3382 (1990)

E. KIERLIK and M. L. ROSINBERG, Phys. Rev. A 44, 5025 (1991)

$$\mathcal{F}_{exc}[\{
ho_i(\mathbf{r})\}] = k_B T \int d\mathbf{r} \, \Phi(\{n_lpha(\mathbf{r})\})$$

$$\begin{split} n_{\alpha}(\mathbf{r}) &= \sum_{i} \int d\mathbf{r}' \,\rho_{i}(\mathbf{r}') \,\omega_{\alpha}^{i}(\mathbf{r} - \mathbf{r}') = \sum_{i} \rho_{i}(\mathbf{r}) \star \omega_{\alpha}^{i}(\mathbf{r}) \\ \omega_{3}^{i}(\mathbf{r}) &= \Theta(R_{i} - r) \qquad \omega_{1}^{i}(\mathbf{r}) &= \frac{1}{8\pi} \delta'(R_{i} - r) \\ \omega_{2}^{i}(\mathbf{r}) &= \delta(R_{i} - r) \qquad \omega_{0}^{i}(\mathbf{r}) &= \frac{1}{8\pi} \,\delta''(R_{i} - r) + \frac{1}{2\pi r} \,\delta'(R_{i} - r) \end{split}$$

$$\Phi^{
m PY}[n_lpha] = -n_0 \ln(1-n_3) + rac{n_1 n_2}{1-n_3} + rac{1}{24\pi} rac{n_2^3}{(1-n_3)^2},$$

#### Fundamental Measure Theory for Hard-Sphere Mixtures



#### Basic theory of liquids: The Ornstein-Zernike equation



## Basic theory of liquids: The Ornstein-Zernike equation

Pair potential 
$$u(r_{12})$$
  
 $h(r_{12}) = g(r_{12}) - 1$   
 $f(\mathbf{r}_{1}) = g(\mathbf{r}_{1}) = g(\mathbf{r}_{1}) = g(\mathbf{r}_{1}) = g(\mathbf{r}_{1}, \mathbf{r}_{2})$   
 $f(\mathbf{r}) = \sum_{i} \delta(\mathbf{r} - \mathbf{R}_{i}), \quad \Delta \hat{\rho}(\mathbf{r}) = \hat{\rho}(\mathbf{r}) - \rho_{0}$   
 $\hat{\rho}(\mathbf{r}) = \sum_{i} \delta(\mathbf{r} - \mathbf{R}_{i}), \quad \Delta \hat{\rho}(\mathbf{r}) = \hat{\rho}(\mathbf{r}) - \rho_{0}$   
 $\chi(\mathbf{r}_{1}, \mathbf{r}_{2}) = \langle \Delta \hat{\rho}(\mathbf{r}_{1}) \Delta \hat{\rho}(\mathbf{r}_{2}) \rangle = \rho_{0} \delta(\mathbf{r}_{1} - \mathbf{r}_{2}) + \rho_{0}^{2} h(\mathbf{r}_{1}, \mathbf{r}_{2})$   
 $\chi^{-1}(\mathbf{r}_{1}, \mathbf{r}_{2}) = \frac{1}{\rho_{0}} \delta(\mathbf{r}_{1} - \mathbf{r}_{2}) - c(\mathbf{r}_{1}, \mathbf{r}_{2})$   
Ornstein-Zernike:  
 $\int d\mathbf{r}_{3} \chi(\mathbf{r}_{1}, \mathbf{r}_{3}) \chi^{-1}(\mathbf{r}_{3}, \mathbf{r}_{2}) = \delta(\mathbf{r}_{1} - \mathbf{r}_{2})$ 

## Basic theory of liquids: The Ornstein-Zernike equation

Pair potential 
$$u(r_{12})$$
  
 $h(r_{12}) = g(r_{12}) - 1$   
 $f(r_{12}) = g(r_{12}) - 1$   
 $f(r_{12}) = g(r_{11}, r_{22})$   
 $f(r_{11})\rho(r_{22}) = g(r_{11}, r_{22})$   
 $\frac{\rho_{2}(r_{11}, r_{22})}{\rho(r_{11})\rho(r_{22})} = \rho(r_{2})g(r_{11}, r_{22})$   
 $= \rho(r_{2})(1 + h(r_{11}, r_{22}))$   
 $f(r_{11}) = \sum_{i} \delta(r_{1} - R_{i}), \quad \Delta \hat{\rho}(r_{12}) = \hat{\rho}_{0}\delta(r_{11} - r_{22}) + \rho_{0}^{2}h(r_{12})$   
 $\chi(r_{11}, r_{22}) = \langle \Delta \hat{\rho}(r_{11})\Delta \hat{\rho}(r_{22}) \rangle = \rho_{0}\delta(r_{11} - r_{22}) + \rho_{0}^{2}h(r_{12})$   
 $\chi^{-1}(r_{11}, r_{22}) = \frac{1}{\rho_{0}}\delta(r_{11} - r_{22}) - c(r_{12})$   
Ornstein-Zernike:  
 $\int dr_{3} \chi(r_{11}, r_{32}) \chi^{-1}(r_{31}, r_{22}) = \delta(r_{11} - r_{22})$ 

The exact functional

$$F[\rho(\mathbf{x})] = F_{id}[\rho] + F_{exc}[\rho] + F_{ext}[\rho] \qquad \mathbf{x} = (\mathbf{r}, \mathbf{\Omega})$$

$$F_{id}[\rho] = k_B T \int d\mathbf{x}_1 \left[ \rho(\mathbf{x}_1) \ln\left(\frac{\rho(\mathbf{x}_1)}{\rho_0}\right) - \rho(\mathbf{x}_1) + \rho_0 \right]$$

$$F_{ext}[\rho] = \int d\mathbf{x}_1 \, V_{ext}(\mathbf{x}_1) \, \rho(\mathbf{x}_1)$$

$$ρ(\mathbf{r}, \Omega)$$

$$F_{exc}[\rho] = \frac{1}{2} \iint d\mathbf{x}_1 d\mathbf{x}_2 \ \Delta \rho(\mathbf{x}_1) \ u_{eff}(\mathbf{x}_1, \mathbf{x}_2; [\rho]) \ \Delta \rho(\mathbf{x}_2), \qquad \Delta \rho(\mathbf{x}) = \rho(\mathbf{x}) - \rho_0$$

 $u_{eff}(\mathbf{x}_1, \mathbf{x}_2; [\rho]) =$  Effective solvent-solvent pair potential

$$u_{eff}(\mathbf{x}_1, \mathbf{x}_2; [\rho]) = u(\mathbf{x}_1, \mathbf{x}_2) + u_{cor}(\mathbf{x}_1, \mathbf{x}_2; [\rho])$$

MEAN-FIELD APPROXIMATION

$$u_{eff}(\mathbf{r}_1, \mathbf{r}_2; [\rho]) = u(r_{12})$$

Adiabatic connection methods

With respect to pair potential:

$$u_{\alpha}(r_{12}) = \alpha \ u(r_{12})$$

$$u_{eff}\left(\mathbf{r}_{1},\mathbf{r}_{2};\left[\rho\right]\right)=\int_{0}^{1}d\alpha h_{\alpha}\left(\mathbf{r}_{1},\mathbf{r}_{2};u_{\alpha}\right)u\left(r_{12}\right)$$

Same as in electronic DFT

With respect to density:  $\rho_{\alpha}(\mathbf{x}) = \rho_0 + \alpha \Delta \rho(\mathbf{x})$ 

$$\frac{1}{2}u_{eff}\left(\mathbf{r}_{1},\mathbf{r}_{2};\left[\rho\right]\right)=k_{B}T\int_{0}^{1}d\alpha\left(\alpha-1\right)c_{\alpha}\left(\mathbf{r}_{1},\mathbf{r}_{2};\left[\rho_{\alpha}\right]\right)$$

#### • MEAN-FIELD APPROXIMATION:

$$u_{eff}(\mathbf{r}_1,\mathbf{r}_2;[\rho]) = u(r_{12})$$
  $u_{cor}(\mathbf{r}_1,\mathbf{r}_2;[\rho]) = 0$ 

#### • HOMOGENEOUS REFERENCE FLUID APPROXIMATION

$$u_{eff}(\mathbf{r}_{1},\mathbf{r}_{2};[\rho]) = -k_{B}T c(\mathbf{r}_{1},\mathbf{r}_{2};\rho_{0}) = -k_{B}T c(r_{12})$$

 $c(\mathbf{r}_{12})$  is the direct correlation function of the homogeneous fluid connected to the pair correlation function  $h(\mathbf{r}_{12})$  through the Ornstein-Zernike relation

$$h(r_{12}) - c(r_{12}) = \rho_0 \int d\mathbf{r}_3 c(r_{13}) h(r_{32})$$



Lennard-Jones fluid: HRF approximation

$$\mathcal{F}[\rho(\mathbf{r})] = k_B T \int d\mathbf{r} \left[ \rho(\mathbf{r}) \ln \left( \frac{\rho(\mathbf{r})}{\rho_0} \right) - \rho(\mathbf{r}) + \rho_0 \right] + \int d\mathbf{r} V(\mathbf{r}) \rho(\mathbf{r}) \\ - \frac{k_B T}{2} \int d\mathbf{r} d\mathbf{r}' \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') c_S(|\mathbf{r} - \mathbf{r}'|; \rho_0) + c_S(|\mathbf{r} - \mathbf{r}'|$$





Lennard-Jones fluid: HRF approximation

$$\mathcal{F}[\rho(\mathbf{r})] = k_B T \int d\mathbf{r} \left[ \rho(\mathbf{r}) \ln \left( \frac{\rho(\mathbf{r})}{\rho_0} \right) - \rho(\mathbf{r}) + \rho_0 \right] + \int d\mathbf{r} V(\mathbf{r}) \rho(\mathbf{r}) \\ - \frac{k_B T}{2} \int d\mathbf{r} d\mathbf{r}' \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') c_S(|\mathbf{r} - \mathbf{r}'|; \rho_0) + c_S(|\mathbf{r} - \mathbf{r}'|$$





Lennard-Jones fluid: HRF + HS bridge approximation

$$\mathcal{F}[\rho(\mathbf{r})] = k_B T \int d\mathbf{r} \left[ \rho(\mathbf{r}) \ln \left( \frac{\rho(\mathbf{r})}{\rho_0} \right) - \rho(\mathbf{r}) + \rho_0 \right] + \int d\mathbf{r} V(\mathbf{r}) \rho(\mathbf{r}) \\ - \frac{k_B T}{2} \int d\mathbf{r} d\mathbf{r}' \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') c_S(|\mathbf{r} - \mathbf{r}'|; \rho_0) + \mathcal{F}_B[\rho(\mathbf{r})]$$

$$\begin{aligned} \mathcal{F}_{B}[\rho(\mathbf{r})] &= \mathcal{F}_{exc}^{HS}[\rho(\mathbf{r})] - \mathcal{F}_{exc}^{HS}[\rho_{0}] - \mu_{exc}^{HS} \int d\mathbf{r} \Delta \rho(\mathbf{r}) \\ &+ \frac{k_{B}T}{2} \int d\mathbf{r} d\mathbf{r}' \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') c_{S}^{HS}(|\mathbf{r} - \mathbf{r}'|; \rho_{0}) \\ &\quad \text{All orders } \Delta \rho^{n}, n \geq 3 \end{aligned}$$





Hard-sphere reference functional: Water

$$\mathcal{F}[\rho(\mathbf{r})] = k_B T \int d\mathbf{r} \left[ \rho(\mathbf{r}) \ln \left( \frac{\rho(\mathbf{r})}{\rho_0} \right) - \rho(\mathbf{r}) + \rho_0 \right] + \int d\mathbf{r} V(\mathbf{r}) \rho(\mathbf{r})$$
  
$$- \frac{k_B T}{2} \int d\mathbf{r} d\mathbf{r}' \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') c_S(|\mathbf{r} - \mathbf{r}'|; \rho_0) + \mathcal{F}_B[\rho(\mathbf{r})]$$
  
$$\mathcal{F}_B[\rho(\mathbf{r})] = \mathcal{F}_{exc}^{HS}[\rho(\mathbf{r})] - \mathcal{F}_{exc}^{HS}[\rho_0] - \mu_{exc}^{HS} \int d\mathbf{r} \Delta \rho(\mathbf{r})$$
  
$$+ \frac{k_B T}{2} \int d\mathbf{r} d\mathbf{r}' \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') c_S^{HS}(|\mathbf{r} - \mathbf{r}'|; \rho_0)$$
  
$$\int \left( \int_{\mathbb{S}_0}^{0} \int_{0}^{0} \int_{0}^{1} \int_{0$$

#### Solvation in SPC water: Alcanes



M. Levesque et al, J. Chem. Phys. 137, 034115, 2012

## Solvation in SPCE water



### Application to solvation of **pyrophyllite** (model neutral clay)



Collaboration with V. Marry and B. Rotenberg (UPMC, Paris) M. Levesque et al, J. Chem. Phys. **137**, 224107, 2012





#### Application to solvation of **pyrophyllite** (model neutral clay)



#### Application to solvation of **pyrophyllite**



# Conclusions

- In analogy to electronic DFT, one can compute solvation free energies, reorganization energies .... and microscopic solvation profiles using classical cDFT (MDFT)
- The functional can be minimized using 3D-grids for positions and angular grids for orientations.
- Good representation of correlations for simple (polar) liquids.
   Much harder for associated liquids such as water
- BEYOND:
  - -- Water and ionic solutions
  - -- Solvent mixtures, ....
  - -- eDFT/MDFT mixing

R. Ramirez et al, Phys. Rev E 66, 2002

J. Phys. Chem. B 114, 2005
Chem. Phys.319, 2005

L. Gendre at al, Chem. Phys. Lett. 474, 2009
S. Zhao et al, J. Chem. Phys. 134, 2011
D. Borgis et al, J. Phys. Chem B 116, 2012
M. Levesque et al, J. Chem. Phys.137, 2012
G. Jeanmairet et al, J. Phys. Chem. Lett. 2013
G. Jeanmairet et al, J. Chem. Phys. 2013